Random Matrix Approach: Toward Probabilistic Formulation of the Manipulator Jacobian

In this paper, we formulate the manipulator Jacobian matrix in a probabilistic framework based on the random matrix theory (RMT). Due to the limited available information on the system fluctuations, the parametric approaches often prove to be inadequate to appropriately characterize the uncertainty. To overcome this difficulty, we develop two RMT-based probabilistic models for the Jacobian matrix to provide systematic frameworks that facilitate the uncertainty quantification in a variety of complex robotic systems. One of the models is built upon direct implementation of the maximum entropy principle that results in a Wishart random perturbation matrix. In the other probabilistic model, the Jacobian matrix is assumed to have a matrix-variate Gaussian distribution with known mean. The covariance matrix of the Gaussian distribution is obtained at every time point by maximizing a Shannon entropy measure (subject to Jacobian norm and covariance positive semidefiniteness constraints). In contrast to random variable/vector based schemes, the benefits of the proposed approach now include: (i) incorporating the kinematic configuration and complexity in the probabilistic formulation; (ii) achieving the uncertainty model using limited available information; (iii) taking into account the working configuration of the robotic systems in characterization of the uncertainty; and (iv) realizing a faster simulation process. A case study of a 2R serial manipulator is presented to highlight the critical aspects of the process. [DOI: 10.1115/1.4027871]

1 Introduction

Kinematic analysis and motion planning in robotic manipulators have traditionally been examined via deterministic methods. However, there are several sources of the uncertainties that can substantially affect the overall performance of the robot in the real world setting. These variations arise from inherently uncertain environment, failure in accurate actuation, sensing uncertainties, modeling inaccuracy, and computational limitations.

Research in motion and kinematic uncertainties in the robotic systems dates as far back as work by Smith and Cheeseman [1] in 1986. They developed a first order model for the propagation of the uncertainty under a series of the frame transformations, given the statistics of the uncertainty associated with a single transformation. Following their work, several researchers performed different studies that are mainly focused on the simultaneous localization and map building [2–5] in the mobile robotic systems. The problem of the uncertainty propagation focusing on the manipulator systems has been addressed in the works by Wang and Chirikjian [6,7]. Their work concerns the error propagation problem on the Euclidean motion groups, in which the distal frame probability density function (pdf) is derived using the convolution of the densities corresponding to each unit along the serial chain. In another effort [7], they extended their method to the second order approximation of larger error propagation using the theory of Lie algebras and Lie groups. Despite the body of literature on the statistics of the uncertainty associated with a single transformation of the densities corresponding to each unit along the serial chain. In another effort [7], they extended their method to the second order approximation of larger error propagation using the theory of Lie algebras and Lie groups. Despite the body of literature on the propagation of the uncertainty under a series of the frame transformations, given the statistics of the uncertainty associated with a single transformation.

The main objective of this paper is to develop a systematic framework to characterize the uncertainty with a focus on robotic manipulator systems. Key requirements for an appropriate uncertainty model for these systems include:

• Taking into account the whole system complexity, structural interdependencies and intercorrelations.
• Considering the variation of the uncertainty characteristics respect to the change in the state of the system.
• Accounting for the system configurations that alter the statistics of the system response, e.g., close to the singularity configurations.
• Applicability when there is limited available information on the uncertainties in the complex systems.
• Computational efficiency.

The novelty of our work arises from formulating the Jacobian matrix of the manipulator as a random matrix. The Jacobian matrix includes information about the kinematic complexity and configuration of the manipulator as well as the information on the singularity regions in which the system shows higher sensitivity to the uncertainties. Moreover, the proposed approach appropriately characterizes the uncertainty in the complex systems with limited available information. It will be shown in the subsequent sections that a scalar parameter (that represents the level of the uncertainty) and a matrix-valued parameter (that in some sense is representative of deterministic Jacobian matrix) are the only information used to construct the matrix-variate probability models.

Conventional parametric approaches are often inadequate to capture the system response variations, especially when dealing with the complex systems. There may exist a large number of statistically dependent parameters whose probabilistic characterization often becomes experimentally and computationally demanding. In addition, in Monte Carlo simulations, the proposed random Jacobian matrix formulation eliminates solving several nonlinear equations resulting in faster simulation process. Hence, the probability model based on the random Jacobian matrix proves to be an appropriate candidate for uncertainty characterization in several complex robotic systems.

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In addition to the physics and statistics, RMT has recently been applied to many engineering applications. In electrical engineering and wireless communications, RMT plays an important role in analyzing multi-input multi-output channels performance [8–12]. Furthermore, RMT has been extensively utilized in several recent works by Soize [13–17], Soize and Chebli [18], Adhikari [19,20], Adhikari et al. [21–24], and Das and Ghanem [25,26] for uncertainty characterization in the multiscale mechanics and vibration problems.

In this paper, based on the random matrix formulation, we develop two probability models for the manipulator Jacobian matrix allowing for uncertainty characterization in a variety of the complex robotic systems. The first approach is based on the model proposed by Soize [13,14,18] for structural system matrices. In this model, the Jacobian matrix adopts a random perturbation matrix whose mean and certain normalized standard deviation of its norm are known. The density function of the perturbation matrix is derived using maximum entropy principle that yields a Wishart distribution. In the second approach, we assume that the Jacobian matrix at every time point of the simulation is perturbed with an additive Gaussian noise matrix with zero mean. To obtain the covariance matrix, another random matrix is defined as a function of the Gaussian noise matrix. Subsequently, the covariance matrix at every time point is obtained by solving an optimization problem in which the Shannon entropy of the new random matrix density function is maximized. We define specific Jacobian norm inequality constraint to be satisfied along with the positive semidefiniteness of the covariance matrix in the optimization problem. To the best of our knowledge, this is the first time that RMT is employed to model the kinematic Jacobian matrix of the uncertain manipulator systems.

Note that the focus of this study is characterization of uncertainty in the joint configuration and is not the problem of error accumulation along the serial chains where the variation of the end effector (EE) response is studied. Having the joint configuration pdf, the distribution of the EE pose can be obtained by the propagation of the frame densities along the kinematic chain [6,7].

The rest of this paper is organized as follows. In Sec. 2, a basic background in manipulators forward and inverse kinematics is reviewed. In Sec. 3, the probabilistic formulations of the Jacobian matrix using RMT are developed. In this section, we first present the preliminaries in the RMT required for our subsequent formulations. Afterward, both the random matrix formulations of the Jacobian matrix are developed. This section concludes with a discussion on the physical justification of the derived formulations. In Sec. 4, a case study of 2R serial chain based on the proposed methods is presented. Finally, a brief discussion and directions of future work are provided in Sec. 5.

2 Forward and Inverse Kinematics

A routine problem in manipulator kinematic analysis is to express the EE position and orientation as a function of the joint space variables (forward kinematics) and vice versa (inverse kinematics). In forthcoming equations, bold letters indicate random entities and matrices are denoted by capital letters. The forward kinematic relation may be stated as

\[ x(t) = g(q(t)) \]  

where \( x \) is the vector of the position and orientation of the EE, \( q \) is the vector of joint variables, \( t \) is time, and \( g \) is a vector function that is, in general, nonlinear and maps from joint space to operational space.

Given the trajectory of the EE, the inverse of function \( g \), if exists, can be used to obtain the joint variables as a function of time. However, in many cases \( g^{-1} \) does not exist or is not explicitly available and requires high level of algebraic and geometric intuitions. Thus, numerical methods, which are applicable to all kinematic structures, are exploited.

The numerical scheme used in this paper employs the differential kinematic relation described as

\[ \dot{x}(t) = J(q(t))\dot{q}(t) \]  

where \( J(q(t)) \) is Jacobian matrix of the kinematic structure. Note that for notational simplicity, the time variable \( t \) is suppressed from the subsequent equations and \( J \) implies \( J(q(t)) \). By assuming that \( J \) is nonsingular, from Eq. (2) one can write

\[ \dot{q} = J^{-1}\ddot{x} \]  

that defines inverse differential kinematic relation in which \( \ddot{x} \) is the desired EE velocity vector. Then, joint variables vector, \( q \), can be numerically obtained in a discrete time domain as

\[ q_{k+1} = q_k + \dot{q}_k\Delta t \quad \text{and} \quad k = 0, \ldots, m \]  

where \( \dot{q}_k \) implies \( q(t_k) \) and \( t_{k+1} = t_k + \Delta t \). Substituting Eq. (3) in Eq. (4) gives

\[ q_{k+1} = q_k + J_k^{-1}\dot{x}_k\Delta t \]  

Thus, given the EE trajectory, \( \dot{x}_k \), and the Jacobian of the manipulator, \( J \), the joint variables can be determined in the discrete time points, \( t_k \)'s. Following the discussion in Sec. 1, the Jacobian matrix in Eq. (5) \((J_k)\) can be considered as a random matrix. Section 3 establishes probabilistic formulations of the Jacobian matrix using RMT.

3 Random Jacobian Matrix

Dealing with a random system, the inverse differential kinematic Eq. (3) can be considered as a general stochastic differential equation given by

\[ \frac{dq}{dt} = f(q, \omega, t) \]  

where \( \omega \) is the process noise. In the discrete time domain, Eq. (6) can be treated as a stochastic difference equation. In special case of the additive independent noise, one can write

\[ q_{k+1} = f(q_k) + \omega_{k+1} = q_k + J(q_k)^{-1}\dot{x}_k\Delta t + \omega_{k+1} \]  

However, due to the facts discussed above, we consider the stochastic difference equation as

\[ q_{k+1} = q_k + J_k^{-1}\dot{x}_k\Delta t \]  

where the disturbing process noise is modeled through the randomness of the Jacobian matrix, given the value of \( q_k \) (in a system without process noise at \( t_{k+1} \), the Jacobian matrix given \( q_k \) is deterministic). As mentioned above, this motion perturbation can be induced by several factors including disturbing forces applied to the manipulator from the uncertain environment, inaccurate sensing, and actuation at the joints, deflection and bending of the links, flexibility of the joints and bases of the manipulator, etc. Hence, given the current configuration \((q_k = q_k)\), the subsequent configuration of the system \((q_{k+1})\) is not deterministically predictable but can be characterized in a probabilistic sense.

In this work, we are in fact interested in characterizing \( p(J_k|q_k = q_k) \), that is the conditional matrix-variate density function of \( J_k \), given \( q_k = q_k \). In construction of this probability model, we mainly use the conditional mean of the Jacobian matrix given the current state of the system (simply called mean of the Jacobian matrix) denoted by \( \bar{J} \), i.e., \( \bar{J} = E[J_k|q_k = q_k] \). In addition, we assume that \( J_k = \bar{J} \). Moreover, in each model, different scalar parameter/s are introduced that represent the level of uncertainty.
Before detailed derivation of the probability models for \( \mathbf{J}_k \), we review some preliminaries in the RMT, required for our subsequent developments.

### 3.1 Preliminaries

In this subsection, some preliminaries in RMT that are necessary for our derivations are presented as definitions and theorems. All the theorems and definitions are from Ref. [27].

**Definition 1.** The \( p \times q \) random matrix \( \mathbf{X} \) is said to have a matrix-variate Gaussian distribution with mean matrix \( \mathbf{M}(p \times q) \) and covariance matrix \( \mathbf{\Sigma} \otimes \mathbf{\Psi} \) where \( \mathbf{Z}(p \times p) \) is positive definite matrix (A is positive definite matrix) and \( \mathbf{\Psi}(q \times q) > 0 \), denoted as \( \mathbf{X} \sim \mathcal{N}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi}) \), if vec \((\mathbf{X}^T) \sim \mathcal{N}(\text{vec} \((\mathbf{M}^T)), \mathbf{\Sigma} \otimes \mathbf{\Psi}) \).

In Definition 1,

\[
\text{vec}(\mathbf{A}_{p \times q}) = \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}
\]

where \( a_i, i = 1, \ldots, q \) is the \( i \)-th column of \( \mathbf{A} \). In addition, \( \otimes \) represents Kronecker products defined by

\[
\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1q}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2q}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}\mathbf{B} & a_{p2}\mathbf{B} & \cdots & a_{pq}\mathbf{B} \end{bmatrix}
\]

The density function of \( \mathbf{X} \) is then given by

\[
p_X(\mathbf{X}) = (2\pi)^{-\frac{1}{2}\delta} |\mathbf{\Sigma}|^{-\frac{1}{2}} |\mathbf{\Psi}|^{-\frac{1}{2}} \exp \left(-\frac{1}{2} \mathbf{\Sigma}^{-1}(\mathbf{X} - \mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X} - \mathbf{M})^T\right)
\]

**Definition 2.** A \( p \times p \) random symmetric positive definite matrix \( \mathbf{S} \), denoted as \( \mathbf{S} \in \mathbb{M}_p^+ \) or \( \mathbf{S} > 0 \), is said to have a Wishart distribution with parameters \( \mathbf{d} \) and \( \mathbf{\Sigma} \in \mathbb{M}_p^+ \), denoted as \( \mathbf{S} \sim \mathcal{W}_p(d, \mathbf{\Sigma}) \), if its pdf is given by

\[
p_S(\mathbf{S}) = \left\{ \frac{2^{dp} \Gamma_p \left( \frac{1}{2} \mathbf{d} \right) |\mathbf{\Sigma}|^{d/2}}{\Gamma_p(d/2)} \right\}^{-1} |\mathbf{S}|^{(d-p-1)/2} \exp \left(-\frac{1}{2} \mathbf{\Sigma}^{-1}\mathbf{S}\right)
\]

where \( \mathbf{S} \in \mathbb{M}_p^+ \), \( d \geq p \)

**Theorem 1.** Let \( \mathbf{X} \sim \mathcal{N}_p(p, \mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi}) \), then \( \mathbf{X}^T \sim \mathcal{N}_p(M^T, \mathbf{\Sigma} \otimes \mathbf{\Psi}) \).

**Theorem 2.** Let \( \mathbf{X} \sim \mathcal{N}_p(p, \mathbf{0}, \mathbf{\Sigma} \otimes \mathbf{I}_q) \), \( q \geq p \), then \( XX^T > 0 \) with probability one.

**Theorem 3.** Let \( \mathbf{X} \sim \mathcal{N}_p(0, \mathbf{\Sigma} \otimes \mathbf{I}_q) \) and define \( \mathbf{S} = XX^T \), \( q \geq p \), then \( \mathbf{S} \sim \mathcal{W}_p(d, \mathbf{\Sigma}) \).

**Theorem 4.** Let \( \mathbf{S} \sim \mathcal{W}_p(d, \mathbf{\Sigma}) \), then \( E[\mathbf{S}] = d\mathbf{\Sigma} \).

### 3.2 Maximum Entropy Formulation of the Jacobian Perturbation Matrix

Before modeling the random Jacobian matrix, we first refer to a method proposed by Soize [13,14] to construct the random symmetric positive definite matrices of the structural systems. The \( n \times n \) random symmetric and positive definite matrix \( \mathbf{A} \in \mathbb{M}_n^+ \) is written as

\[
\mathbf{A} = L^T \mathbf{G}_a L \quad (13)
\]

where \( L \) is an upper triangular matrix corresponding to Cholesky decomposition of \( \mathbf{A} \), which is the mean of random matrix \( \mathbf{A} \), and \( \mathbf{G}_a \) is a random symmetric positive definite matrix \( \mathbf{G}_a \in \mathbb{M}_n^+ \) with identity mean.

Although the Jacobian matrix is, in general, rectangular and can also be singular, in several common cases the number of joint space variables is identical to the number of EE degrees of freedom, i.e., the Jacobian matrix is square. In addition, the task of the manipulator is commonly defined such that the manipulator does not reach the singularity regions. Then, the Jacobian matrix is nonsingular during the operation time. Thus, let us assume the Jacobian matrix of the kinematic structure is a nonsingular square matrix with dimensions \( n, \mathbf{J}_k \in \mathbb{M}_n^+ \). In a similar fashion, we decompose the mean of the Jacobian matrix as

\[
\mathbf{J}_k = L_{J_k} U_{J_k}
\]

Equation (14) is LU decomposition of Jacobian mean matrix in which \( U_{J_k} \) is an upper triangular matrix and \( L_{J_k} \) is such that \( L_{J_k} U_{J_k} = \mathbf{J}_k \). Now, for the random Jacobian matrix we can write

\[
\mathbf{J}_k = L_{J_k} \mathbf{B} U_{J_k}
\]

where \( \mathbf{B} \in \mathbb{M}_n^+ \); in fact, Eq. (15) states that random perturbing matrix \( \mathbf{B} \) is symmetric and positive definite although the Jacobian matrix is not in general. From Eq. (15), the expected value of the random matrix \( \mathbf{B} \) can be obtained as follows:

\[
E[\mathbf{J}_k] = E[L_{J_k} \mathbf{B} U_{J_k}] = L_{J_k} E[\mathbf{B}] U_{J_k}
\]

Equating the right hand side of Eqs. (14) and (16) gives

\[
E[\mathbf{B}] = I_n
\]

Thus, the available information and constraint on the density function of the random matrix \( \mathbf{B} \) are

\[
\int_{\mathbf{B} \in \mathbb{M}_n^+} p_B(\mathbf{B}) d\mathbf{B} = 1
\]

\[
E[\mathbf{B}] = \int_{\mathbf{B} \in \mathbb{M}_n^+} \mathbf{B} p_B(\mathbf{B}) d\mathbf{B} = \mathbf{B} = I_n
\]

Equation (18) normalizes \( p_B(\mathbf{B}) \).

To find the appropriate matrix-variate distribution, an information-theoretic approach is employed. In this approach, all information about the matrix as well as its physical constraints are first considered. The density function is obtained using maximum entropy principle proposed by Jaynes [28] in which the entropy measure, initially introduced by Shannon [29], is maximized subject to the existing constraints. In fact, the pdf derived through this method provides maximum uncertainty while satisfying the constraints.

Maximum entropy (MaxEnt) method [30] has been frequently utilized to obtain the density function of the random variates using the known information. Soize [13,14,18], Adhikari [19], and Ghanem and Das [31] used the MaxEnt method to derive the probability model of the structural system matrices (mass, damping, stiffness, and frequency response function matrices) when the mean system is known. Das and Ghanem [25,26] further addressed this problem incorporating additional upper and lower bound constraints on the system matrices. This allows for appropriate constructions of less uncertain probability models when bounding information are available.

In the first approach, described in this subsection, a procedure similar to the existing approaches mentioned above is followed to find the matrix-variate density function of random matrix \( \mathbf{B} \).

Entropy measure for density function \( p_B(\mathbf{B}) \) is defined as [28,30]

\[
S(p) = - \int_{\mathbf{B} \in \mathbb{M}_n^+} p_B(\mathbf{B}) \ln[p_B(\mathbf{B})] d\mathbf{B}
\]

Lagrangian multiplier method can be used to maximize \( S(p) \) subjected to the constraints in Eqs. (19) and (18). The Lagrangian functional is constructed as

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\[ L(p_B(B)) = - \int_{B \in \mathcal{M}_2^n} p_B(B) \ln p_B(B) dB \]
\[ - (\lambda_0 - 1) \left( \int_{B \in \mathcal{M}_2^n} p_B(B) dB - 1 \right) \]
\[ - \text{tr} \left[ \Lambda_1 \left( \int_{B \in \mathcal{M}_2^n} B p_B(B) dB - B \right) \right] \]  
\( (21) \)

where \( \lambda_0 - 1 \in \mathbb{R} \) and \( \Lambda_1 \in \mathcal{M}_2^n \) are Lagrange multipliers corresponding to the normalization and mean constraints, respectively. Using the variational and matrix calculus, it can be shown that (see Ref. [19])
\[ p_B(B) = \frac{\rho^{|B|^2}}{\Gamma_d(r)} e^{tr(-rB^{-1}B)} \text{ and } r = \frac{1}{2} (n + 1) \]  
\( (22) \)

Comparing Eq. (22) with Eq. (12) (Definition 2) shows that \( B \) has a Wishart distribution with parameters \( d = n + 1 \) and \( \Sigma = (B / (\theta + n + 1)) \) where \( \theta = 2\gamma \).

From the construction of the pdf of \( B \) only \( \theta \) remains undetermined. To this end, the normalized standard deviation of \( B \) is defined as [13,19]
\[ \sigma_B^2 = \frac{E[\|B - E[B]\|_F^2]}{\|E[B]\|_F^2} \]  
\( (23) \)

Here, \( \theta \) can be derived from Eq. (24) as [19]
\[ \theta = \frac{1}{\sigma_B^2} \left( 1 + \frac{tr(B^2)}{tr(B)} \right) - (n + 1) \]  
\( (25) \)

in which \( \sigma_B \) is called dispersion parameter of random matrix \( B \) and a measure of uncertainty that was initially introduced by Soize [13]. This value can be assumed from experience or experimental measurements. For example, one may use accurate sensors to observe the state (configuration) of the system at different time points (with specific sampling rate) for several tests. Let us denote the true (measured) state of the system at \( k \)th step in \( n \)th test by \( \tilde{q}_k \). Now, the \( n \)th realization of the process noise can be simply obtained by
\[ a_{k+1} = q_{k+1} - \tilde{q}_{k+1} = q_k + J(\tilde{q}_k)^{-1} \tilde{x}_k \Delta t - \tilde{q}_{k+1} \]  
\( (26) \)

Now, from the model proposed in Eq. (8) the process noise added to the state at \( t_{k+1} \) can be obtained by
\[ a_{k+1} = (J_k^{-1} - L_k^{-1}) \tilde{x}_k \Delta t \]  
\( (27) \)

Hence, given \( \tilde{q}_k \) (from measurement) and \( \tilde{q}_{k+1} \) from Eq. (26) and substituting \( J_k \) from Eq. (15), we have
\[ (L_k^{-1} B_{k+1} L_k U_k)^{-1} - L_k^{-1}) \tilde{x}_k \Delta t = a_{k+1} \]  
\( (28) \)

Note that, as mentioned above, we assume the (conditional) mean of Jacobian matrix at \( t_k \) to be the Jacobian obtained by substituting given state at time \( t_k \), i.e., \( J_k = J(\tilde{q}_k) \). In case of experiment, when accurate states are available, one may write \( J_k = J(q_k) \). Now, \( i \)th realization of the perturbation matrix \( B \) at time \( t_{k+1} \) (i.e., \( B_{k+1} \)) can be obtained by minimizing the following objective function subject to positive definiteness of \( B_{k+1} \):
\[ F = \|a_{k+1} - (L_k^{-1} B_{k+1} U_k)^{-1} - L_k^{-1}) \tilde{x}_k \Delta t\| + \|B_{k+1} - I\| \]  
\( (29) \)

The penalty term in Eq. (29) is due to the fact that the mean of \( B \) is identity matrix for all \( t_k \)’s. Once the realizations of the perturbation matrix \( B \) are obtained, the dispersion parameter can be calculated using Eq. (24).

In the Monte Carlo simulation, having Wishart distribution parameters, samples of random matrix \( B \) can be generated using existing Wishart sampling algorithms (see, for example, Ref. [31]). Here, we use \texttt{matlab} function \texttt{wishrnd} to draw samples of \( B \). \texttt{MATLAB} automatically converts \( d \) parameter value to the nearest integer for noninteger values of \( \theta \) obtained from Eq. (25). Then, the samples of the random Jacobian matrix can be obtained by substituting the samples of \( B \) in Eq. (15). Algorithm 1 summarizes the procedure for Monte Carlo simulation of \( J_k \) and subsequently \( q_{k+1} \).

**Algorithm 1:** Monte Carlo simulation of \( J_k \) 
**Inputs:** \( q_k, \sigma_B, \) and \( n \); **Output:** Sample of \( J_k(\tilde{J}_k) \) and sample of \( q_{k+1}(\tilde{q}_{k+1}) \)

1. Substitute \( q_k \) in the \( J(\tilde{q}_k) \) to obtain \( \tilde{J}_k \)
2. Calculate \( \theta \) from Eq. (25)
3. Set \( d = \theta + n + 1 \) and \( \Sigma = (B / (\theta + n + 1)) \)
4. Generate a sample \( \tilde{B} \) of \( B \) (\texttt{MATLAB}’s command \texttt{wishrnd} may be used here)
5. Calculate \( L_k \) and \( U_k \) from LU decomposition of \( \tilde{J}_k \)
6. Substitute \( \tilde{L}_k \), \( \tilde{J}_k \), and \( \tilde{U}_k \) in Eq. (15) to obtain a sample \( \tilde{J}_k \) of \( J_k \)
7. Substitute \( \tilde{J}_k \) in Eq. (8) to obtain a sample of \( q_{k+1} \)

### 3.3 Gaussian Jacobian Matrix Formulation

In this subsection, we propose an alternative probabilistic model for the formulation of the random Jacobian matrix. Here, norm of the Jacobian matrix (as a measure of manipulability) is also incorporated in the probabilistic formulation. In addition, it directly provides the density of Jacobian matrix while the model based on Wishart perturbation matrix gives the density of the perturbation matrix rather than Jacobian itself. Here, we assume that the mean Jacobian matrix (at every time point) is perturbed with an additive Gaussian noise matrix. An optimization problem is then solved to calculate the covariance matrix. The objective function is defined to be Shannon entropy of the pdf of a function of the noise matrix. Along with the positive semidefiniteness of the covariance matrix, we also define a specific Jacobian norm inequality constraint to be satisfied in the optimization problem.

We assume that \( J_k \) is perturbed with an additive Gaussian noise matrix \( J_{nk} \sim N_{nk}(0, I \otimes \Sigma_k) \). So, the random Jacobian matrix \( J_k \) is
\[ J_k = \tilde{J}_k + J_{nk} \]
\[ \Rightarrow J_k \sim N_{nk}(\tilde{J}_k, I \otimes \Sigma_k) \]  
\( (30) \)

\( I \otimes \Sigma_k \) is the covariance matrix of the \( J_k \) (and \( J_{nk} \)) and implies that the rows of the Jacobian matrix are independent random vectors. This can be inferred from the physical interpretation of the manipulator differential kinematic relation, given by Eq. (3). Equation (3) describes that every row of the Jacobian matrix corresponds to one component of the EE velocity. Assuming that the velocity components of the manipulator EE are independent random variables entails the independence of the rows of the Jacobian matrix.

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Now, let us define $Y = J_q^aJ_d$. Using Theorems 1 and 3, $Y \sim W_d(n, \Sigma)$. The objective is to find $\Sigma$ such that the entropy of the $\rho(Y)$ is maximized.

The Shannon entropy of the Wishart density function $p_{\Sigma}(\Sigma) = 1/M^n_{\Sigma}$ with the state of the system ($\Sigma$) and the Jacobian matrix are either constant quantities or vary. Proceeding to the optimization problem, we review some features of the manipulator Jacobian matrix. The components of the manipulator Jacobian matrix have bounded Frobenius norm. Let us now consider the entropy of the Wishart density function $S(p_{\Sigma}(\Sigma)) = -\ln B(\Sigma, d) - d - p - 1/2 \ln|\Sigma| + dp/2$.

where

\begin{equation}
B(\Sigma, d) = |\Sigma|^{-1}\left(\pi^{n/2} / \sqrt{n} \Gamma(n/2 + d/2 + 1)\right)^{d/2}
\end{equation}

\begin{equation}
E[\ln \Sigma] = \sum_{i=1}^{p} \psi\left(\frac{d + 1 - i}{2}\right) + p \ln 2 + \ln |\Sigma|
\end{equation}

in which $\Gamma(\cdot)$ is the univariate gamma function and $\psi(\cdot)$ is the digamma function. Substituting Eqs. (32) and (33) in Eq. (31) yields

\begin{equation}
S(p_{\Sigma}(\Sigma)) = c_1 \ln |\Sigma| + c_2
\end{equation}

where

\begin{equation}
c_1 = \frac{p + 1}{2}
\end{equation}

\begin{equation}
c_2 = \frac{1}{2} p(p + 1) \ln 2 + \frac{1}{2} p(p - 1) \ln \pi + \sum_{i=1}^{d} \ln \left(\Gamma\left(\frac{d + 1 - i}{2}\right)\right)
\end{equation}

\begin{equation}
\frac{1}{2} (d - p - 1) \sum_{i=1}^{p} \psi\left(\frac{1}{2} (d + 1 - i)\right) + dp/2
\end{equation}

Here, we have $Y \sim W_d(n, \Sigma)$. So, in Eq. (34) $p = n = d$. Before proceeding to the optimization problem, we review some features of the manipulator Jacobian matrix. The components of the manipulator Jacobian matrix are either constant quantities or vary with the state of the system ($q$). The state-varying components are trigonometric functions of the joint variables ($q_i$)'s. This results in a Jacobian matrix with bounded Frobenius norm. Let us now assume that the upper bound of the Frobenius norm of the $J_k$ is known. So, we have

\begin{equation}
||J||_F \leq u \Rightarrow ||J||_F^2 \leq u^2
\end{equation}

Equation (35) also implies

\begin{equation}
E[||J||_F^2] \leq u^2
\end{equation}

From Eq. (30) and given that $||A||_F = \sqrt{\text{tr}(AA^T)}$, we can write $||J||_F^2$ as

\begin{equation}
||J||_F^2 = ||J_d + J_{sk}||_F^2 = \text{tr}([J_d + J_{sk}]^2 + J_{sk}^2)
\end{equation}

Using the linearity of the $\text{tr}(.)$ operator and given $\text{tr}(A) = \text{tr}(A^T)$ and $\text{tr}(AB) = \text{tr}(BA)$, we can then write Eq. (37) as

\begin{equation}
||J||_F^2 = \text{tr}(J_d^2 + J_{sk}^2) + 2 \text{tr}(J_d J_{sk}) + \text{tr}(J_{sk}^2)
\end{equation}

Now, from inequality (36) together with Eq. (38) and using the linearity of the mean and $\text{tr}(.)$ operator we have

\begin{equation}
\text{tr}(E[J_d^2 + J_{sk}^2]) + 2 \text{tr}(E[J_d J_{sk}]) \leq u^2 - ||J_d||_F^2
\end{equation}

In relation (39), substituting $Y = J_d^2 J_{sk}$ and $E[J_d J_{sk}^2] = E[J_{sk} J_d^2] = 0$ gives

\begin{equation}
\text{tr}(E[Y]) \leq u^2 - ||J_d||_F^2
\end{equation}

Given that $Y \sim W_d(n, \Sigma)$ (Theorem 3), then using Theorem 4, $E[Y] = n \Sigma$. So, we finally get

\begin{equation}
\text{tr}(n \Sigma) \leq u^2 - ||J_d||_F^2
\end{equation}

In Eq. (34), $c_1$ and $c_2$ can be ignored in the cost function as they are independent of the $\Sigma$ and $c_1 > 0$. So, the cost function for maximizing the entropy $S(p_{\Sigma}(\Sigma))$ can simply be $f = \ln |\Sigma|$. In addition, the optimization constraints are the Jacobian norm inequality described in Eq. (41) along with the positive semidefiniteness of $\Sigma$. However, applying only norm inequality constraint described by inequality (41) may yield a maximally uncertain distribution. To avoid unrealistic uncertainty in the resulting distribution, we multiply the right hand side of the inequality (41) by factor $\alpha$, where $\alpha$ is a positive constant parameter. Finally, the matrix optimization problem turns out to be

\begin{equation}
\text{Maximize } f = \ln |\Sigma|_1
\end{equation}

Subject to

\begin{equation}
\text{tr}(n \Sigma) \leq \alpha^2 (n^2 - ||J_d||_F^2)
\end{equation}

\begin{equation}
\Sigma \geq 0
\end{equation}

Here, $\alpha$ can be determined based on the experiment or prior knowledge of the system. For example, in a similar fashion to $\sigma_B$ characterization, once the values of $\Sigma$’s are determined from a set of experimental measurements, the best estimate of the parameter $\alpha$, satisfying inequality (43), may be readily obtained. Algorithm 2 summarizes the Monte Carlo simulation based on the Gaussian Jacobian matrix model.

Algorithm 2: Monte Carlo simulation based on Gaussian Jacobian matrix model. Inputs: $q_k$, $n$, and $u$. Outputs: $\Sigma_k$, $J_k$, and $\hat{q}_{k+1}$.

1. Substitute $q_k$ in the $J(q_k)$ to obtain $\bar{J}_k$
2. Find $||\bar{J}_k||_F$.
3. Obtain $\Sigma_k$ by solving Eqs. (42)–(44)
4. Given $\bar{J}_k$ and $\Sigma_k$, generate $J_k$ from matrix-variate normal distribution (MATLAB's command mvnrnd may be useful here)
5. Substitute $J_k$ in Eq. (8) to obtain a sample of $q_{k+1}$

3.4 Physical Justifications. Robotic manipulators are more sensitive to the disturbances and show more fluctuations while working in the regions close to the singularity or when their kinematic manipulability degrades. The kinematic manipulability of the robotic manipulator refers to its ability in moving or reorienting the EE in different directions. This changes with the joint configuration of the robot during the task performance. To quantify the manipulability, different measures, and indices have been proposed based on the Jacobian matrix of the manipulator. Yoshikawa manipulability measure, defined in Eq. (45), is one of the most commonly used measures in the robotic studies

\begin{equation}
W = \sqrt{\text{det}(JJ^T)}
\end{equation}

Another manipulability measure is the weighted (squared) Frobenius norm of the Jacobian matrix, given by [33]

\begin{equation}
\Phi = \frac{1}{n} \text{tr}(JJ^T)
\end{equation}

where $n$ is the workspace dimension. From Eqs. (45) and (46), it can be seen that Yoshikawa measure has information about the singularity regions; however, the weighted Frobenius norm does not provide this information.
Referring to Eq. (15), when the manipulator configuration is close to the singular configurations or when, based on Yoshikawa measure, it loses its kinematic manipulability, the variation in $J_1^{-1}$ (that determines the variation in the response $q_0$) increases. This is induced by the decrease in the determinant of the Jacobian. This also holds in the Gaussian Jacobian matrix formulation, as we have modeled $J_1$ instead of $J_1^{-1}$. One may consider $J_1^{-1}$ as random matrix in similar formulation and assumes $J_1^{-1}$ as known information. However, the resulting model does not take into account the singularity regions and the change of the (Yoshikawa) manipulability measure. In addition to this feature of both models, there is a noteworthy fact associated with the model based on the Gaussian Jacobian matrix formulation. In fact the covariance adapting rule described in inequality (43) uses the Frobenius norm instead of $J_1$. This also holds in the Gaussian Jacobian matrix formulation, as the covariance of the resulting noise affects the covariance of the resulting noise (that determines the variation in the response). But, for the Jacobian matrix that is, based on Eq. (46), a measure of the kinematic manipulability. In addition to this feature of both models, the Jacobian determinant (that determines the variation in the response $q_0$) reduces in both these regions; however, the Jacobian Frobenius norm measure reduces only around the base of the manipulator (because it does not provide information about the singularity regions). As depicted in Fig. 1(a), the determinant is defined such that the manipulator passes the regions near the robot base (close to the singularity), in which both the manipulability measures decreases.

4 Numerical Illustration: 2R Planar Serial Manipulator (Double Pendulum)

In this section, we implement both developed methods (Wishart perturbation matrix and Gaussian Jacobian matrix) for uncertainty characterization in a 2R planar serial chain, shown in Fig. 1(a), with two identical links ($a = 1.5$ m). A trajectory, described in Eqs. (47) and (48), is first defined to be followed by the manipulator EE such that the robot experiences both high-manipulability (start and end of the path) and close to singularity configurations (middle of the path). The total time of the simulation is 10 s and $\Delta t = 0.005$ s.

$$x_{ee}^d(t) = 0.3t - 1.5 \quad (47)$$
$$y_{ee}^d(t) = 0.04t^2 - 0.4t + 1.5 \quad (48)$$

In Fig. 1(b), the contour plot illustrates Yoshikawa manipulability corresponding to each EE position. There are two close to singularity regions, one near the base of the manipulator and one near the workspace boundary. The difference is that Yoshikawa measure reduces in both these regions; however, the Jacobian Frobenius norm measure reduces only around the base of the manipulator (because it does not provide information about the singularity regions). As depicted in Fig. 1(b), the trajectory is defined such that the manipulator passes the regions near the robot base (close to the singularity), in which both the manipulability measures decreases.

4.1 Simulation Based on the Wishart Perturbation. Using Algorithm 1, assuming $\sigma_B = 0.25$ (for all time points) and $q_0 \sim N(q_0, 1 \times 10^{-3} \times I_{3 \times 3})$, we performed a Monte Carlo analysis with 1000 simulations of the defined trajectory tracking task. Figure 2 shows the standard deviation of the $\omega_0$ and $\omega_0$ versus the time of the simulation (where realizations of $\omega_0 = [\omega_x, \omega_y, \omega_z]^T$ are obtained from Eq. (27)).

The determinant of the Jacobian, the EE velocities ($v_{ee}^d$ and $v_{ee}^u$), and joint rates ($\dot{\theta}_1$ and $\dot{\theta}_2$) for the deterministic (nominal) manipulator are also shown in Fig. 2 (right axis). It is clear from Eq. (27) that the EE velocities also affect the standard deviation of the $\omega_0$ and $\omega_0$. As Fig. 2 shows, the standard deviation of the $\omega_0$ raises in the singularity region; however, the standard deviation of the $\omega_0$ reduces. Here, the predefined trajectory and instability induced by working in the singularity region are both affecting the covariance of the resulting noise $\omega_0$ (or standard deviations of $\omega_{0x}$ and $\omega_{0y}$). As discussed before, reduction in the Jacobian determinant amplifies the variation in both noise components. However, as shown in Fig. 2, the trajectory is defined such that $v_{ee}^d$ goes to zero simultaneously when the manipulator approaches the singularity region. This considerably decreases the standard deviation of $\omega_0$ but not $\omega_0$. The reason for this fact is that $\dot{\theta}_2$ also goes to zero simultaneously with $v_{ee}^d$ (in a deterministic manipulator). This is illustrated in Fig. 2, where diamond and circle plots intersect. From deterministic equation

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = J^{-1} \begin{bmatrix} v_{ee}^d \\ v_{ee}^u \end{bmatrix},$$

we can write $\dot{\theta}_2 = J_{12}^{-1} v_{ee}^d + J_{22}^{-1} v_{ee}^u$. Because in our case, $\dot{\theta}_2 \to 0$ when $v_{ee}^d \to 0$, then $J_{12}^{-1} v_{ee}^d \to 0$. Now (given $\theta_2$), $\theta_{x+1} = \theta_2 + J_{12}^{-1} v_{ee}^d + J_{22}^{-1} v_{ee}^u$. Knowing that the last two terms are rapidly going to zero implies the compensation of the large fluctuations in the $J_{12}^{-1}$ and $J_{22}^{-1}$ at this time point/s. However, this is not the case for $J_{22}^{-1}$ and $J_{12}^{-1}$. If the Jacobian determinant is not tending to zero and consequently the effect of large variations in the $J_{12}^{-1}$ (due to the determinant decrease) is not compensated. It can be seen that the trajectory/motion planning in the uncertain robotic manipulators can play an important role in the overall performance of the system that, in turn, necessitates a systematic approach to investigate different aspects of the process.

The joint density function of the $\omega_0$ and $\omega_0$ at every time point ($t_2$) can be estimated using 2D kernel smoothing (KS) function estimation (MATLAB command ksdensity). Figure 3 shows these densities corresponding to four different time points.
(t = 1, 5, 7.5, and 9 s). To illustrate the evolution of the covariance matrix of \( \omega \) over the time, the sample covariance matrices are calculated at several time points and their corresponding ellipsoids are plotted in Fig. 4. For a bivariate Gaussian distribution with mean \( \mu \) and covariance matrix \( \Sigma \), the equal probability contours are defined by

\[ (X - \mu)^T \Sigma^{-1} (X - \mu) = c^2 \]  

(49)

Equation (49) represents an (in general) ellipsoid centered at \( \mu \). Constant \( c \) determines the fraction of the probability mass enclosed by the ellipsoid. Here, \( c \) is chosen such that 90% of the probability mass is enclosed. Note that, in Fig. 4, for illustration purpose, the ellipsoids are not centered at mean, but at the corresponding EE position. In addition, their sizes are magnified with the same factor to be visualized in the manipulator workspace (x–y plane).

The results shown in Fig. 4 verify those in Fig. 2 and additionally illustrate the correlation between \( \omega_{h_1} \) and \( \omega_{h_2} \) at different time points.

To show the propagation of the uncertainty, the evolutions of the kernel density estimates of \( h_1 \) and \( h_2 \) over the time are shown in Fig. 5. In addition, Fig. 6 shows the standard deviation of \( h_1 \) and \( h_2 \) versus the time.

4.2 Simulation Based on Gaussian Jacobian Matrix. Similar to the simulation in Sec. 4.1, for an identical manipulator, task, and initial condition, a Monte Carlo analysis with 1000 simulations is performed based on the Gaussian Jacobian matrix approach using Algorithm 2. The parameter \( z \) is selected to be equal to 0.06 and \( u = 3.5 \). A drawback of this method is that it requires the solution of a nonlinear optimization problem to obtain the noise covariance. One effective treatment to this problem is to update the covariance matrix in every \( N \) time steps (instead of every time step), as the norm of the Jacobian matrix may be not rapidly changing in several tasks/systems. However, for systems with higher speed or tasks close to the singularity regions, the value of \( N \) may be adaptively selected in different speeds/regions.

Similarly, the standard deviation of \( h_1 \) and \( h_2 \) versus the time is plotted in Fig. 7. Comparing to the results of Wishart
perturbation matrix approach, depicted in Fig. 2, this model shows similar trend of the noise standard deviations against the time. However, in this approach, the change (increase) of the standard deviations in the areas close to the base of the manipulator (close to the singularity regions) with respect to the high-manipulability areas is more significant than those in Fig. 2 (Wishart perturbation matrix approach). As discussed before, the reason behind this fact is that Gaussian Jacobian matrix formulation incorporates both manipulability measures (Yoshikawa and Jacobian norm) that are decreasing in the areas near the base. However, Wishart perturbation matrix approach takes into account only the effect of Yoshikawa measure.

Figure 8 shows the ellipsoids corresponding to $\Sigma$ calculated at different time points ($t_k$'s) along with the Jacobian norm contour in the manipulator workspace. The ellipsoids are centered at the EE position corresponding to the time point in which $\Sigma$ is calculated. As the Jacobian, in this case, is a $2 \times 2$ matrix, to maximize the entropy measure that is $\ln|\Sigma|$, the off diagonal term of $\Sigma(2 \times 2)$ has to be zero. This can be seen in Fig. 8 where the (generally) ellipsoids of the $\Sigma_k$'s are converted to the circles. The increase in the circle sizes when the EE approaches the robot base illustrates the increase of the adapting noise standard deviations governed by inequality (43).

Figure 9 illustrates the KS estimation of the joint density function of the $x_{h_1}$ and $x_{h_2}$ (at $t = 1, 5, 7.5, \text{and } 9$ s) obtained based on the Gaussian Jacobian matrix approach.

Similar to Sec. 4.1, the evolutions of the kernel density estimates of $\theta_1$ and $\theta_2$ over the time and the variation of $\theta_1$ and $\theta_2$ standard deviations versus the time are shown in Figs. 10 and 11, respectively.

Indeed, the resulting propagation of the uncertainty in the joint state variables (here are $\theta_1$ and $\theta_2$) depends on the uncertainty characterization models as well as the predefined task (trajectory) that acts as an input to the system. Here, we examined the proposed uncertainty characterization methods and investigated the propagation of the uncertainty (based on these methods) for a trajectory tracking task in an open-loop kinematic control simulation. Characterizing the uncertainty and subsequently investigating its propagation in a given system, in fact, provide a...
systematic framework facilitating the design of the systems that are more robust to the uncertainties.

5 Discussion

In this paper, for the first time the RMT was employed to formulate the Jacobian matrix of the random manipulators that provides a probabilistic framework to study the effects of the uncertainties in the kinematic systems. Two uncertainty approaches were proposed to model the Jacobian matrix. In the first approach, based on a model proposed by Soize [13,14,18], Jacobian matrix is treated as random and its matrix-variate density function was derived through direct implementation of the maximum entropy method. In the second approach, we assumed that the mean Jacobian matrix was perturbed by an additive and zero mean Gaussian noise matrix. An optimization problem was constructed to obtain the covariance of the noise matrix (or covariance of the random Jacobian matrix) in which the Shannon entropy of the pdf of a function of the noise matrix is maximized. The constraint of the optimization problem was constructed based on the boundedness of the Jacobian norm, allowing for adapting the noise covariance matrix based on the working configuration of the manipulator system. The positive semidefiniteness of the covariance matrix was also required to be satisfied in the optimization problem. The physical justification of the both proposed models was argued and different critical aspects were discussed. While simulation of the Gaussian random matrix is more efficient, no nonlinear optimization is required for the Wishart-based model.

A case study of 2R serial chain (double pendulum) was performed to highlight different aspects of the simulation process. A specific trajectory tracking task was defined for the manipulator EE, and the characteristics of the perturbing noise, formulated based on both proposed methods, were separately discussed and compared. It was shown that both methods show similar trends in the uncertainty characterization and propagation. However, there are differences that make them better applicable to different problems depending on the existing conditions.

Uncertainty characterization based on the RMT, in general, allows to consider the kinematic structure and complexity of the robotic systems in the probabilistic formulation, resulting in the models that better represent the real uncertain systems. In addition, it facilitates the construction of an appropriate probability model when there is only limited available information about the system uncertainty, especially in the complex robotic systems.
Moreover, the system matrices have information on the system state and working configuration (e.g., singularity regions) and the random matrix based probability models permit incorporating these information in the uncertainty characterization process. Furthermore, random matrix based Monte Carlo simulation is computationally more efficient as it eliminates solving many nonlinear equations.

Although the effectiveness of these models is analytically justified, experimental studies are required to draw a general conclusion on the accuracy of the proposed models. Incorporating the real experimental results with the proposed frameworks can be beneficial in appropriate modification to the model-parameters or real experimental results with the proposed frameworks can be beneficial in appropriate modification to the model-parameters.

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References


